THREE CLASSES OF LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING GAMMA AND PSI FUNCTIONS

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Abstract. By a simple approach, two classes of functions involving Euler's gamma function and originating from certain problems of traffic flow are proved to be logarithmically completely monotonic and a class of functions involving the psi function is showed to be completely monotonic.

1. Introduction

Recall [12, 22, 23] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $0 \le (-1)^n f^{(n)}(x) < \infty$ for $x \in I$ and $n \ge 0$. The set of the completely monotonic functions on I is denoted by $\mathcal{C}[I]$. The well known Bernstein's Theorem [23, p. 161] states that $f \in \mathcal{C}[(0,\infty)]$ if and only if $f(x) = \int_0^\infty e^{-xs} \,\mathrm{d}\mu(s)$, where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$. This expresses that $f \in \mathcal{C}[(0,\infty)]$ if and only if f is a Laplace transform of the measure μ.

Recall [1, 6, 12, 15, 17, 19, 20] also that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0 \le (-1)^k[\ln f(x)]^{(k)} < \infty$ for all $k \in \mathbb{N}$ on I. The set of the logarithmically completely monotonic functions on I is denoted by $\mathcal{L}[I]$. In [2, Theorem 1.1] and [6, 20] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [7, Theorem 4.4].

It was proved in [2, 12, 19, 20, 22] that $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely. Stimulated by the papers [17, 19], among other things, it was further revealed in [2] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)]$, where $\mathcal{S}$ denotes the set of Stieltjes transforms.

The Kershaw's inequality in [8] states that the double inequality

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\right)^{1-s} \tag{1}$$

holds for $0 < s < 1$ and $x \ge 1$, where Γ denotes the classical Euler's gamma function and $\psi = \frac{\Gamma'}{\Gamma}$, the logarithmic derivative of Γ, the psi function. If taking $s = \frac{1}{2}$ in (1),

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then, for $x>1$,

$$\sqrt{x+\frac{1}{4}}<\frac{\Gamma(x+1)}{\Gamma(x+1/2)}<\sqrt{x+\frac{\sqrt{3}-1}{2}}\,. \tag{2}$$

Let s and t be nonnegative numbers and $\alpha=\min\{s,t\}$. For $x\in(-\alpha,\infty)$, define

$$z_{s,t}(x)=\begin{cases}\left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}-x, & s\ne t,\\ e^{\psi(x+s)}-x, & s=t.\end{cases} \tag{3}$$

In order to establish the best bounds in Kershaw's inequality (1), among other things, the papers [3, 5, 13, 18] established the following monotonicity and convexity property of $z_{s,t}(x)$: The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s|<1$ or concave and increasing for $|t-s|>1$. This result was further generalized in the papers [10, 11].

In [4, p. 123] and [9], while studying certain problems of traffic flow, a double inequality below was obtained for $n\in\mathbb{N}$:

$$2\Gamma\biggl(n+\frac12\biggr)\le\Gamma\biggl(\frac12\biggr)\Gamma(n+1)\le2^n\Gamma\biggl(n+\frac12\biggr), \tag{4}$$

In [21], inequality (4) was extended and refined for $x>0$ as

$$\sqrt{x}\le\frac{\Gamma(x+1)}{\Gamma(x+1/2)}\le\sqrt{x+\frac12}. \tag{5}$$

It is clear that the double inequality (5) is weaker than (2).

Observe that inequality (4) can be rearranged for $n>1$ as

$$1\le\left[\frac{\Gamma(1/2)\Gamma(n+1)}{2\Gamma(n+1/2)}\right]^{1/(n-1)}\le2. \tag{6}$$

Hinted by this, the following function $g(x)$ was defined in [14] for $x\in\bigl(-\frac12,\infty\bigr)$:

$$g(x)=\begin{cases}\left[\dfrac{\Gamma(1/2)\Gamma(x+1)}{2\Gamma(x+1/2)}\right]^{1/(x-1)}, & x\ne1,\\ \exp\biggl[1-\gamma-\psi\biggl(\dfrac32\biggr)\biggr], & x=1,\end{cases} \tag{7}$$

where $\gamma=0.57721566\cdots$ is Euler-Mascheroni's constant, and, among other things, it was proved in [14] that the function $g(x)$ is logarithmically complete monotonic in $\bigl(-\frac12,\infty\bigr)$: $g(x)\in\mathcal{L}\bigl[\bigl(-\frac12,\infty\bigr)\bigr]$ with $\lim_{x\to-1/2}g(x)=\infty$ and $\lim_{x\to\infty}g(x)=1$. As consequences of this result, it is deduced that

$$2\Gamma\biggl(x+\frac12\biggr)\le\Gamma\biggl(\frac12\biggr)\Gamma(x+1)\le2\Gamma\biggl(x+\frac12\biggr)\exp\biggl\{(x-1)\biggl[1-\gamma-\psi\biggl(\frac32\biggr)\biggr]\biggr\} \tag{8}$$

for $x\in[1,\infty)$ and

$$2\Gamma\biggl(x+\frac12\biggr)\le\Gamma\biggl(\frac12\biggr)\Gamma(x+1)\le2^x\Gamma\biggl(x+\frac12\biggr) \tag{9}$$

for $x\in(0,\infty)$. It was remarked in [14] that inequalities (8) and (9) extend (4) and (6), the right hand side inequality of (8) refines the right hand side inequality of

(4) and (6), and the right hand side inequalities in (2) and (5) and the following inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \le \frac{2}{\Gamma(1/2)} \exp\left\{(x-1)\left[1-\gamma-\psi\left(\frac{3}{2}\right)\right]\right\} \tag{10}$$

for $x \in [1,\infty)$, which is deduced from the right hand side inequality of (8), are not included with each other respectively.

Now rewrite inequality (4) or (6) for $n > 1$ as

$$1 \le \left[\frac{\Gamma(1+1/2)}{\Gamma(1+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+1/2)}\right]^{1/(n-1)} \le 2. \tag{11}$$

The definition (7) of $g(x)$ and inequality (11) motivate us to introduce a new function $h_\beta(x)$ as follows: Let s and t be nonnegative numbers with $s \ne t$, $\alpha = \min\{s,t\}$ and $\beta > -\alpha$. For $x \in (-\alpha,\infty)$, define

$$h_\beta(x) = \begin{cases} \left[\dfrac{\Gamma(\beta+t)}{\Gamma(\beta+s)} \cdot \dfrac{\Gamma(x+s)}{\Gamma(x+t)}\right]^{1/(x-\beta)}, & x \ne \beta, \\ \exp[\psi(\beta+s)-\psi(\beta+t)], & x = \beta. \end{cases} \tag{12}$$

The first aim of this paper is to consider the logarithmically completely monotonic property of $h_\beta(x)$ by a simple approach. Our first main result is the following Theorem 1.

Theorem 1. (1) *If $s > t$, then $h_\beta(x) \in \mathcal{L}[(-\alpha,\infty)]$ with*

$$\lim_{x\to-\alpha} h_\beta(x) = \infty \qquad \text{and} \qquad \lim_{x\to\infty} h_\beta(x) = 1. \tag{13}$$

(2) *If $s < t$, then $[h_\beta(x)]^{-1} \in \mathcal{L}[(-\alpha,\infty)]$ with*

$$\lim_{x\to-\alpha} h_\beta(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} h_\beta(x) = 1. \tag{14}$$

Remark 1. It is noted that taking $s = 1$, $t = \alpha = \frac{1}{2}$ and $\beta = 1$ in Theorem 1 can deduce one of the results obtained in [14], the logarithmically complete monotonicity of the function $g(x)$ defined by (7).

Since $h_\beta(x)$ is decreasing (or increasing) for $s > t$ (or $s < t$), $h_\beta(\beta) = \exp[\psi(\beta+s)-\psi(\beta+t)]$ and $\lim_{x\to\infty} h_\beta(x) = 1$, then the following double inequality (15), as a direct consequence of Theorem 1, is established easily.

Corollary 1. *Let s and t be nonnegative numbers, $\alpha = \min\{s,t\}$ and $\beta > -\alpha$. If $s > t$, inequality*

$$\frac{\Gamma(\beta+s)}{\Gamma(\beta+t)} \le \frac{\Gamma(x+s)}{\Gamma(x+t)} \le \frac{\Gamma(\beta+s)}{\Gamma(\beta+t)} \exp\{(x-\beta)[\psi(\beta+s)]-\psi(\beta+t)\} \tag{15}$$

holds for $x \in [\beta,\infty)$. If $s < t$, inequality (15) reverses.

Remark 2. If taking $\beta = 1$, $s = 1$ and $t = \frac{1}{2}$ in (15), then inequality (8) is deduced. So, it can be said that inequality (15) is a generalization of (8).

In [16], it was showed that the function $\ln x - \frac{1}{2x} - \psi(x) \in \mathcal{C}[(0,\infty)]$. In order to prove Theorem 3 below, this result need to be generalized. Our second main result is the following Theorem 2.

Theorem 2. *Let $\alpha \in \mathbb{R}$. Then $\psi(x) - \ln x + \frac{\alpha}{x} \in \mathcal{C}[(0,\infty)]$ if and only if $\alpha \ge 1$ and $\ln x - \frac{\alpha}{x} - \psi(x) \in \mathcal{C}[(0,\infty)]$ if and only if $\alpha \le \frac{1}{2}$.*

For $x \in (0,\infty)$, define

$$p(x)=\begin{cases}\left[\dfrac{x^x}{\Gamma(x+1)}\right]^{1/(1-x)}, & x\neq 1,\\ e^{-\gamma}, & x=1.\end{cases} \tag{16}$$

In [14], among other things, the logarithmically complete monotonicity of $p(x)$ was proved: $p(x)\in\mathcal{L}[(0,\infty)]$ with $\lim_{x\to 0+}p(x)=1$ and $\lim_{x\to\infty}p(x)=\frac{1}{e}$. Motivated by inequality (11) and the definition of $h_\beta(x)$ in (12), a more general function than $p(x)$ can be introduced: For $x\in(0,\infty)$ and $\alpha>0$, let

$$p_\alpha(x)=\begin{cases}\left[\dfrac{\Gamma(\alpha+1)}{\alpha^\alpha}\cdot\dfrac{x^x}{\Gamma(x+1)}\right]^{1/(\alpha-x)}, & x\neq\alpha,\\ \dfrac{\exp[\psi(\alpha+1)-1]}{\alpha}, & x=\alpha.\end{cases} \tag{17}$$

It is clear that $p_1(x)=p(x)$.

The third aim of this paper is to show the logarithmically completely monotonicity of the function $p_\alpha(x)$ for any fixed $\alpha>0$ by a simple approach. Our third main result is the following Theorem 3.

Theorem 3. *For any fixed $\alpha>0$, $p_\alpha(x)\in\mathcal{L}[(0,\infty)]$ with*

$$\lim_{x\to 0+}p_\alpha(x)=\frac{\sqrt[\alpha]{\Gamma(\alpha+1)}}{\alpha}\qquad\text{and}\qquad\lim_{x\to\infty}p_\alpha(x)=\frac{1}{e}. \tag{18}$$

2. Proofs of theorems

It is well-known (see [12, 15, 17, 18, 19, 20]) that, for $x>0$ and $\omega>0$,

$$\frac{1}{x^\omega}=\frac{1}{\Gamma(\omega)}\int_0^\infty t^{\omega-1}e^{-xt}\,\mathrm{d}\,t, \tag{19}$$

and that, for $k\in\mathbb{N}$ and $x>0$,

$$\psi(x)=\ln x+\int_0^\infty\left(\frac{1}{u}-\frac{1}{1-e^{-u}}\right)e^{-xu}\,\mathrm{d}\,u, \tag{20}$$

$$\psi^{(k)}(x)=(-1)^{k+1}\int_0^\infty\frac{t^ke^{-xt}}{1-e^{-t}}\,\mathrm{d}\,t. \tag{21}$$

Proof of Theorem 1. Without loss of generality, assume $s>t$. For $x\neq\beta$, taking logarithm of the function $h_\beta(x)$ gives

$$\begin{aligned}\ln h_\beta(x)&=\frac{1}{x-\beta}\left[\ln\frac{\Gamma(x+s)}{\Gamma(\beta+s)}-\ln\frac{\Gamma(x+t)}{\Gamma(\beta+t)}\right]\\&=\frac{\ln\Gamma(x+s)-\ln\Gamma(\beta+s)}{x-\beta}-\frac{\ln\Gamma(x+t)-\ln\Gamma(\beta+t)}{x-\beta}\\&=\frac{1}{x-\beta}\int_\beta^x\psi(u+s)\,\mathrm{d}\,u-\frac{1}{x-\beta}\int_\beta^x\psi(u+t)\,\mathrm{d}\,u\\&=\frac{1}{x-\beta}\int_\beta^x[\psi(u+s)-\psi(u+t)]\,\mathrm{d}\,u\\&=\frac{1}{x-\beta}\int_\beta^x\int_t^s\psi'(u+v)\,\mathrm{d}\,v\,\mathrm{d}\,u\end{aligned}$$

$$\triangleq \frac{1}{x-\beta}\int_\beta^x \Phi_{s,t}(u)\,\mathrm{d}\,u$$
$$= \int_0^1 \Phi_{s,t}((x-\beta)u+\beta)\,\mathrm{d}\,u$$

and, by differentiating $\ln h_\beta(x)$ with respect to x,

$$[\ln h_\beta(x)]^{(k)} = \int_0^1 u^k \Phi_{s,t}^{(k)}((x-\beta)u+\beta)\,\mathrm{d}\,u \tag{22}$$

for $k \in \mathbb{N}$.

If $x = \beta$, formula (22) is also valid.

Formula (21) implies that $\psi' \in \mathcal{C}[(0,\infty)]$ and $\Phi_{s,t}(u) \in \mathcal{C}[(-t,\infty)]$. This means that $(-1)^i[\Phi_{s,t}(u)]^{(i)} \ge 0$ holds in $u \in (-t,\infty)$ for any nonnegative integer i. Thus,

$$(-1)^k[\ln h_\beta(x)]^{(k)} = \int_0^1 u^k\big\{(-1)^k \Phi_{s,t}^{(k)}((x-\beta)u+\beta)\big\}\,\mathrm{d}\,u \ge 0 \tag{23}$$

in $(-t,\infty)$ for $k \in \mathbb{N}$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. Formulas (19) and (20) imply that

$$\psi(x) - \ln x + \frac{\alpha}{x} = \int_0^\infty \left[\frac{(1-u)e^u - 1}{u(e^u-1)} + \alpha\right] e^{-xu}\,\mathrm{d}\,u \triangleq \int_0^\infty [\theta(u)+\alpha]e^{-xu}\,\mathrm{d}\,u,$$
$$\theta'(u) = \frac{(u^2+2)e^u - 1 - e^{2u}}{u^2(e^u-1)^2} \triangleq \frac{\theta_1(u)}{u^2(e^u-1)^2},$$
$$\theta_1'(u) = [u^2+2u+2-2e^u]e^u \triangleq e^u\theta_2(u),$$
$$\theta_2'(u) = 2(1+u-e^u) < 0.$$

It is clear that the function $\theta_2(u)$ is decreasing in $(0,\infty)$. From $\theta_2(0)=0$, it follows that $\theta_2(u)<0$ and $\theta_1'(u)<0$ in $(0,\infty)$ and that $\theta_1'(u)$ decreases in $(0,\infty)$. Since $\theta_1'(0)=0$, then $\theta_1'(u)<0$ and $\theta_1(u)$ is decreasing in $(0,\infty)$. From $\theta_1(0)=0$, it follows that $\theta_1(u)<0$ and $\theta'(u)<0$ in $(0,\infty)$, and then the function $\theta(u)$ is decreasing in $(0,\infty)$. L'Hôspital's rule yields that $\lim_{u\to0+}\theta(u) = -\frac12$ and $\lim_{u\to\infty}\theta(u) = -1$. Thus,

$$\theta(u)+\alpha \begin{cases} \ge 0, & \alpha \ge 1 \\ \le 0, & \alpha \le \dfrac12 \end{cases}$$

on $[0,\infty]$. Since $\big[\psi(x)-\ln x+\frac{\alpha}{x}\big]^{(k)} = (-1)^k\int_0^\infty[\theta(u)+\alpha]u^k e^{-xu}\,\mathrm{d}\,u$ for nonnegative integer k, then $\psi(x)-\ln x+\frac{\alpha}{x} \in \mathcal{C}[(0,\infty)]$ if $\alpha \ge 1$ (or $\ln x - \frac{\alpha}{x} - \psi(x) \in \mathcal{C}[(0,\infty)]$ if $\alpha \le \frac12$).

If $\psi(x)-\ln x+\frac{\alpha}{x} \in \mathcal{C}[(0,\infty)]$ (or $\ln x-\frac{\alpha}{x}-\psi(x) \in \mathcal{C}[(0,\infty)]$), then $\psi(x)-\ln x+\frac{\alpha}{x} > 0$ (or <0), this leads to $\alpha > x[\ln x - \psi(x)]$ (or $\alpha < x[\ln x-\psi(x)]$) in $(0,\infty)$. Since $\lim_{x\to0+}\{x[\ln x-\psi(x)]\} = 1$ (or $\lim_{x\to\infty}\{x[\ln x-\psi(x)]\} = \frac12$), then $\alpha \ge 1$ (or $\alpha \le \frac12$). The proof of Theorem 2 is complete. □

Proof of Theorem 3. From the well known differences equation $\Gamma(x+1) = x\Gamma(x)$, it follows easily that

$$\psi(x+1) - \psi(x) = \frac{1}{x} \tag{24}$$

for $x > 0$. For $x \neq \alpha$, taking logarithm of $p_\alpha(x)$ and using (24) gives

$$\begin{aligned}
\ln p_\alpha(x) &= \frac{1}{x-\alpha}\left[\ln\frac{\Gamma(x+1)}{\Gamma(\alpha+1)} - \frac{x^x}{\alpha^\alpha}\right] \\
&= \frac{\ln\Gamma(x+1)-\ln\Gamma(\alpha+1)}{x-\alpha} - \frac{x\ln x - \alpha\ln\alpha}{x-\alpha} \\
&= \frac{1}{x-\alpha}\int_\alpha^x \psi(u+1)\,\mathrm{d}\,u - \frac{1}{x-\alpha}\int_\alpha^x [1+\ln x]\,\mathrm{d}\,u \\
&= \frac{1}{x-\alpha}\int_\alpha^x \left[\psi(u)-\ln u+\frac{1}{u}-1\right]\mathrm{d}\,u \\
&\triangleq \frac{1}{x-\alpha}\int_\alpha^x [\Psi(u)-1]\,\mathrm{d}\,u \\
&= \int_0^1 \Psi((x-\alpha)u+\alpha)\,\mathrm{d}\,u - 1
\end{aligned}$$

and, by differentiating $\ln p_\alpha(x)$ with respect to x,

$$[\ln p_\alpha(x)]^{(k)} = \int_0^1 u^k \Psi^{(k)}((x-\alpha)u+\alpha)\,\mathrm{d}\,u \tag{25}$$

for $k \in \mathbb{N}$.

If $x = \alpha$, formula (25) is also valid.

From Theorem 2, it follows that $\Psi(u) \in \mathcal{C}[(0,\infty)]$. This implies $(-1)^i\Psi^{(i)}(u) \geq 0$ for nonnegative integer i. As a result, for $k \in \mathbb{N}$,

$$(-1)^k[\ln p_\alpha(x)]^{(k)} = \int_0^1 u^k\left\{(-1)^k\Psi^{(k)}((x-\alpha)u+\alpha)\right\}\mathrm{d}\,u \geq 0 \tag{26}$$

holds in $x \in (0,\infty)$. The proof of Theorem 3 is complete. □

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